Exam Questions

1. This question is compulsory.

a) Answer the following questions about probability and random processes.

i) Explain what is meant by a wide-sense stationary random process and what the Wiener-Khinchine theorem says about it.

[3]

ii) Given two statistically independent Gaussian random variables with zeros means and the same variances, how would you generate a Rayleigh random variable and a Ricean random variable?

[4]

iii) Explain what is meant by the term “ergodicity”. Is the sinusoid $X(t) = A\cos(\omega_c t + \Theta)$ with random phase Θ uniformly distributed on $[0, 2\pi]$ ergodic? (There is no justification required.)

[3]

b) Answer the following questions about modulation and demodulation.

i) Explain the terms “synchronous detection”, “envelope detection”, “coherent detection”, and “noncoherent detection”.

[4]

ii) Draw a diagram for the demodulation of single-sideband (SSB) amplitude-modulated signals where the carrier is suppressed. Indicate the bandwidth of the bandpass filter.

[3]

iii) Can the regular phase shift-keying (PSK) signal be noncoherently detected? Explain what is meant by differential phase shift-keying (DPSK).

[3]

c) Answer the following questions about information theory and coding.

i) Explain how Shannon defines and measures information.

[5]

ii) Explain what is meant by mutual information, how channel capacity is defined, and write down the Shannon capacity formula for the additive white Gaussian noise channel.

[5]

d) Answer the following questions about noise.

i) Explain what the term “additive white Gaussian noise” means. Is Gaussian noise always white?

[4]

[Continued on the following page.]

ii) A bandpass noise signal $n(t)$ can be expressed as $n(t) = n_c(t)\cos\omega_c t + n_s(t)\sin\omega_c t$. Consider bandpass noise $n(t)$ having the power spectral density shown below in Fig. 1.1. Draw the power spectral density of $n_s(t)$ if the center frequency $\omega_c/2\pi$ is 8 MHz.

[6]

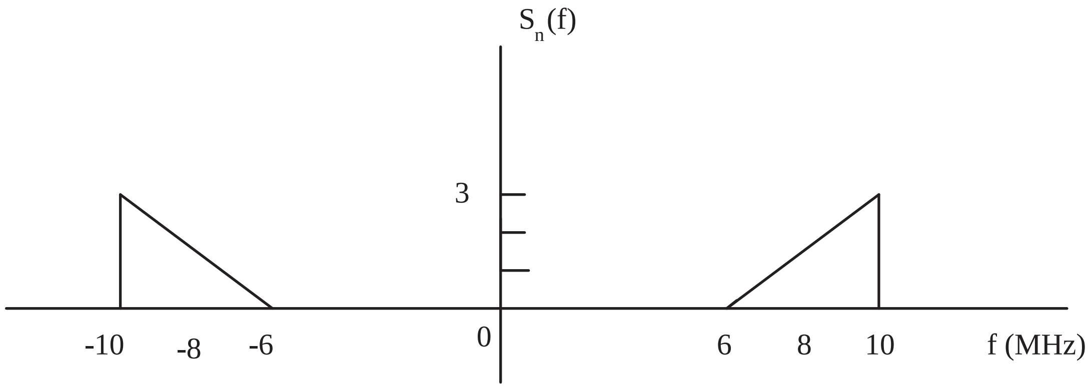


Figure 1.1 Power spectral density of $n(t)$.

2. Analogue communications.

a) A single-sideband (SSB) signal is transmitted over a noisy channel, with the power spectral density of the noise

$$\mathbb{S}(f) = \begin{cases} N_o\left(1 - \frac{|f|}{B}\right), & |f| < B \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

where $B = 200$ kHz and $N_o = 10^{-9}$ W/Hz. The message has bandwidth 10 kHz and average power 10 W. The carrier amplitude at the transmitter is 1 V. Assume the channel attenuates the signal power by a factor of 1000, i.e., 30 decibel (dB). Assume the lower sideband (LSB) is transmitted and a suitable bandpass filter is used at the receiver to limit the out-of-band noise. Determine the predetection SNR at the receiver if

i) the carrier frequency is 100 kHz;

[8]

ii) the carrier frequency is 200 kHz.

[6]

b) In practice, the de-emphasis filter in an FM receiver is often a simple resistance-capacitance (RC) circuit with transfer function

$$H_{de}(f) = \frac{1}{1 + j2\pi fRC} \tag{2.2}$$

i) Calculate the 3-dB bandwidth and equivalent bandwidth.

[4]

ii) Suppose the modulating signal has bandwidth W, the carrier amplitude is A, and the single-sided power spectral density of the white Gaussian noise is N_0. Compute the noise power at the output of the de-emphasis filter.

[6]

iii) Compute the noise power without the de-emphasis filter.

[3]

iv) Now suppose $RC = 6 \times 10^{-5}$, and $W = 15$ kHz. Compute the improvement in the output signal-to-noise ratio (SNR) provided by the de-emphasis filter. Express it in decibel (dB).

[3]

3. Digital communications.

a) A uniform quantizer for PCM has 2^n levels. The input signal is $m(t) = A_m[\cos(\omega_m t) + \sin(\omega_m t)]$. Assume the dynamic range of the quantizer matches that of the input signal.

i) Write down the expressions for the signal power, quantization noise power, and the SNR in dB at the output of the quantizer.

[6]

ii) Determine the value of n such that the output SNR is about 62 dB.

[4]

b) Consider a binary digital modulation system, where the carrier amplitude at the receiver is 1 V, and the white Gaussian noise has standard deviation 0.2. Assume that symbol 0 and symbol 1 occur with equal probabilities.

i) Compute the bit error rates for ASK, FSK, and PSK with coherent detection. Use the following approximation to the Q-function

$$Q(x) \lesssim \frac{1}{\sqrt{2\pi} \cdot x} e^{-x^2/2}, \quad x \geq 0 \tag{3.1}$$

[5]

ii) Compute the bit error rates for ASK, FSK, and DPSK with noncoherent detection.

[5]

c) The Q-function is widely used in performance evaluation of digital communication systems. More precisely, $Q(x)$ is defined as the probability that a standard normal random variable X exceeds the value x :

$$Q(x) \triangleq \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt, \quad x \geq 0 \tag{3.2}$$

i) It is known that $Q(x)$ admits an alternative expression

$$Q(x) = \frac{1}{\pi} \int_0^{\pi/2} e^{-\frac{x^2}{2\sin^2\theta}} d\theta, \quad x \geq 0 \tag{3.3}$$

Using this alternative expression, show the upper bound $Q(x) \leq \frac{1}{2} e^{-x^2/2}$.

[4]

ii) By the definition (3.2), show that (3.1) is an upper bound on $Q(x)$, i.e.,

$$Q(x) \leq \frac{1}{\sqrt{2\pi} \cdot x} e^{-x^2/2}, \quad x \geq 0 \tag{3.4}$$

[Hint: use integration by parts for $e^{-t^2/2}$ in (3.2).]

[6]

4. Information theory and coding.

a) Consider an information source generating the random variable X with probability distribution

x_k	x_1	x_2	x_3	x_4	x_5
$P(X = x_k)$	0.3	0.1	0.15	0.15	0.3

i) Construct a binary Huffman code for this information source. The encoded bits for the symbols should be shown.

[6]

ii) Compute the efficiency η of this code, where the efficiency is defined as the ratio between the entropy and the average codeword length:

$$\eta = \frac{H(X)}{\overline{L}} \tag{4.1}$$

[6]

b) A $(7,4)$ cyclic code has a generator polynomial $g(z) = g_0 z^3 + g_1 z^2 + g_2 z + 1 = z^3 + z^2 + 1$.

i) Write down the generator matrix in the systematic form.

[6]

ii) Find the parity check polynomial associated with this generator polynomial.

[4]

iii) What is the minimum Hamming distance? [Justification is required.] How many errors can this code detect and correct respectively?

[4]

iv) Is this a “perfect” code in the sense of the Hamming bound? [Justification is required.]

[4]

Answers

1. a) i) The mean $E[X(t)]$ of a wide-sense stationary random process doesn't depend on t, and the autocorrelation function $R_X(t_1, t_2)$ depends only on $\tau = t_1 - t_2$. [2, bookwork]

The Wiener-Khinchine theorem says that the power spectral density is the Fourier transform of $R_X(\tau)$. [1, bookwork]

ii) Give two statistically independent Gaussian random variables X and Y, [2, bookwork]

Rayleigh random variable $Z_1 = \sqrt{X^2 + Y^2}$ [2, bookwork]

Ricean random variable $Z_2 = \sqrt{(A+X)^2 + Y^2}$, A a constant

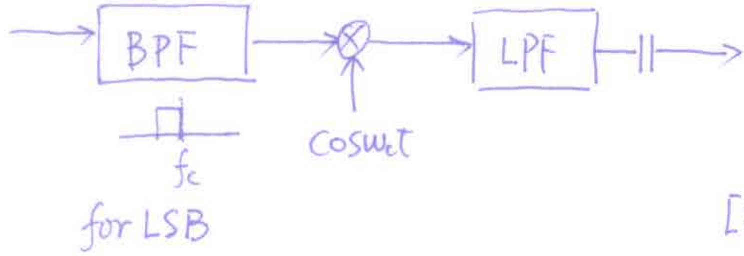
iii) Ergodicity: A wide-sense stationary random process is ergodic if:

- its time average = ensemble average
- time autocorrelation function = ensemble autocorrelation function. [2, bookwork]

Yes, it is ergodic. [1, bookwork]

b) i)
- Synchronous detection: needs a local carrier that is synchronized with the incoming carrier. [1, bookwork]
- envelope detection: tracks the envelope of the signal; no local carrier needed. [1, bookwork]
- Coherent detection: the same as synchronous detection, a term used more often in digital communications. [1, bookwork]
- Noncoheret detection: does not require phase synchronization at the receiver. [1, bookwork]

ii)



[3, bookwork]

iii) NO. [3, bookwork]

In DPSK, the information symbols are differentially encoded, thus permitting differential detection.

c) i) Information of a symbol s: $I(s) = \log_2\left(\frac{1}{p}\right)$

Entropy of an information source $S = \{s_1, s_2, \cdots, s_K\}$

$$H(S) = -\sum_{k=1}^{K} p_k \log_2(p_k) \quad \text{bits/symbol}$$

[5, bookwork]

ii) Mutual information

$$I(X;Y) = H(X) - H(X|Y)$$

Channel capacity

$$C = \max_{p(x)} I(X;Y)$$

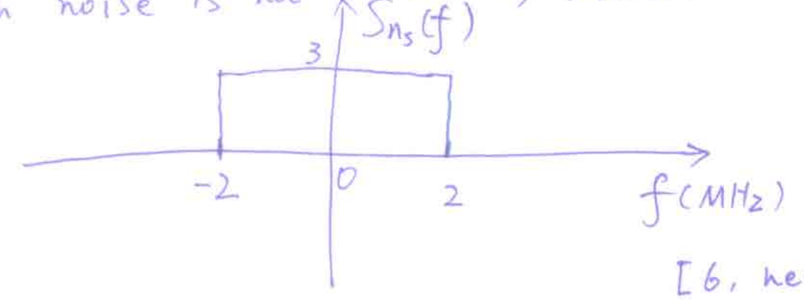
[5, bookwork]

Capacity formula $C = B\log_2\left(1+\frac{S}{N}\right)$

d) i) "Additive white Gaussian noise" means noise is added to the signal, it has Gaussian distribution, and its power spectral density is a constant. [4, bookwork]

Gaussian noise is not necessarily white.

ii)

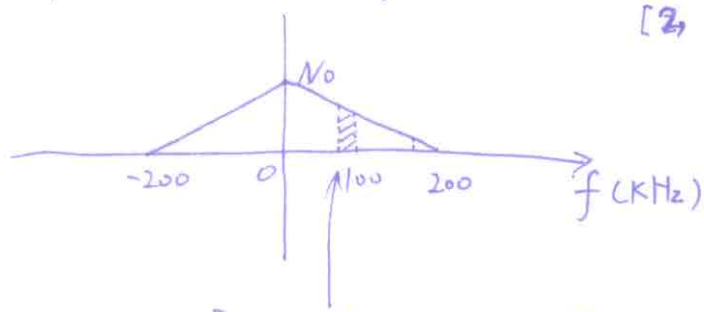


[6, new example]

2. a) i) Transmitted power $P_T = \frac{A^2 P}{4} = 2.5\ W$

Received power $P_R = \frac{2.5}{1000} = 2.5\times10^{-3}\ W = 2.5\ mW$

[2, book work]



Noise power P_N = the area × 2

$$= 2\times\left(\frac{N_0}{2}\times 10\,kHz + \frac{0.05N_0}{2}\times 10\,kHz\right)$$

$$= 10.5\ \mu W$$ [3, new example]

$$SNR = \frac{P_R}{P_N} = \frac{2.5\times10^{-3}}{10.5\times10^{-6}} = 238 \quad (23.8\,dB)$$

[3, new example]

ii) Noise power $P_N = 2\times\frac{0.05N_0}{2}\times 10\,kHz$

$$= 0.5\times10^{-6}\ W$$ [3, new example]

$$= 0.5\ \mu W$$

$$SNR = \frac{2.5\times10^{-3}}{0.5\times10^{-6}} = 5000 \quad (37\,dB)$$

[3, new example]

b) i) 3dB bandwidth

$$|H_{dc}(f_{3dB})| = \frac{1}{\sqrt{1+(2\pi f_{3dB}RC)^2}} = \frac{1}{\sqrt{2}}$$

$$f_{3dB} = \frac{1}{2\pi RC}$$

equivalent bandwidth

$$B_{eq} = \frac{\int_0^\infty |H_{dc}(f)|^2 df}{|H_{dc}(0)|^2} = \int_0^\infty \frac{1}{1+(2\pi fRC)^2}df$$

$$= \frac{1}{2\pi RC}\tan^{-1}(x)\Big|_0^\infty = \frac{\frac{\pi}{2}}{2\pi RC} = \frac{1}{4RC}$$

[4, new example]

ii) After de-emphasis, noise PSD becomes

$$S_D(f) = \frac{f^2}{A^2} N_0 \frac{1}{1+(f/f_{3dB})^2}$$

$$P_N = \int_{-W}^{W} S_D(f)\, df = \frac{N_0}{A^2} \int_{-W}^{W} \frac{f^2}{1+(f/f_{3dB})^2}\, df$$

$$= \frac{N_0}{A^2} f_{3dB}^2 \int_{-W}^{W} \left[1 - \frac{1}{1+(f/f_{3dB})^2}\right] df$$

$$= \frac{N_0}{A^2} f_{3dB}^2 \left[2W - 2f_{3dB} \tan^{-1}\left(\frac{W}{f_{3dB}}\right)\right]$$

$$= 2\frac{N_0}{A^2} f_{3dB}^3 \left[\frac{W}{f_{3dB}} - \tan^{-1}\left(\frac{W}{f_{3dB}}\right)\right]$$

[6, new theory]

iii) Without de-emphasis

$$P_N = \frac{2N_0 W^3}{3A^2}$$

[3, bookwork]

iv) Improvement

$$I = \frac{\dfrac{2N_0W^3}{3A^2}}{2\dfrac{N_0}{A^2} f_{3dB}^2 \left[\dfrac{W}{f_{3dB}} - \tan^{-1}\left(\dfrac{W}{f_{3dB}}\right)\right]}$$

$$= \frac{W^3}{3 f_{3dB}^3 \left[\dfrac{W}{f_{3dB}} - \tan^{-1}\left(\dfrac{W}{f_{3dB}}\right)\right]}$$

$$RC = 6\times10^{-5} \Rightarrow f_{3dB} = \frac{1}{2\pi RC} = \frac{1}{2\pi\times6\times10^{-5}} = 2.65 \text{ kHz}$$

$$I = \frac{15^3}{3\times2.65^3\times\left[\frac{15}{2.65} - 1.4\right]} = \frac{3375}{238} = 14.2$$

(11.5 dB)

[3, new application]

3. a) i) $m(t) = A_m(\cos w_m t + \sin w_m t)$

$$= \sqrt{2} A_m \cos(w_m t - \tfrac{\pi}{4})$$

$$P_S = \tfrac{1}{2}(\sqrt{2}A_m)^2 = A_m^2 \qquad \Delta = \frac{2\sqrt{2}A_m}{2^n}$$

$$P_N = \frac{\Delta^2}{12} = \frac{8A_m^2}{12 \times 2^{2n}} = \frac{2A_m^2}{3 \times 2^{2n}}$$

$$SNR = \frac{P_S}{P_N} = \frac{3}{2} \times 2^{2n} \Rightarrow 6.02n + 1.76 \text{ dB}$$

[6, new example]

ii) $6.02n + 1.76 = 62$

$$n = 10$$

[4, book work]

b) i) $A/\sigma = 1/0.2 = 5$

$$P_{e,ASK} = Q\left(\frac{A}{2\sigma}\right) = Q(2.5) \approx \frac{e^{-2.5^2/2}}{\sqrt{2\pi} \times 2.5} = 7 \times 10^{-3}$$

$$P_{e,FSK} = Q\left(\frac{A}{\sqrt{2}\sigma}\right) = Q\left(\frac{5}{\sqrt{2}}\right) = 2.1 \times 10^{-4}$$

$$P_{e,PSK} = Q\left(\frac{A}{\sigma}\right) = Q(5) = 3 \times 10^{-7}$$

[5, bookwork]

ii) Noncoherent detection

$$P_{e,ASK} = \frac{1}{2} e^{-\frac{A^2}{8\sigma^2}} = 2.2 \times 10^{-2}$$

$$P_{e,FSK} = \frac{1}{2} e^{-\frac{A^2}{4\sigma^2}} = 9.7 \times 10^{-4}$$

$$P_{e,PSK} = \frac{1}{2} e^{-\frac{A^2}{2\sigma^2}} = 1.9 \times 10^{-6}$$

[5, bookwork]

c) i) $$Q(x) = \frac{1}{\pi}\int_0^{\pi/2} e^{-\frac{x^2}{2\sin^2\theta}}\, d\theta$$

$$\le \frac{1}{\pi}\int_0^{\pi/2} e^{-\frac{x^2}{2}}\, d\theta \quad \text{since } \sin^2\theta \le 1$$

$$= \frac{\pi/2}{\pi} e^{-\frac{x^2}{2}}$$

$$= \frac{1}{2} e^{-\frac{x^2}{2}}$$

[4, new theory]

ii) $$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt$$

$$= -\int_x^\infty \frac{1}{\sqrt{2\pi}\,t}\, d e^{-t^2/2}$$

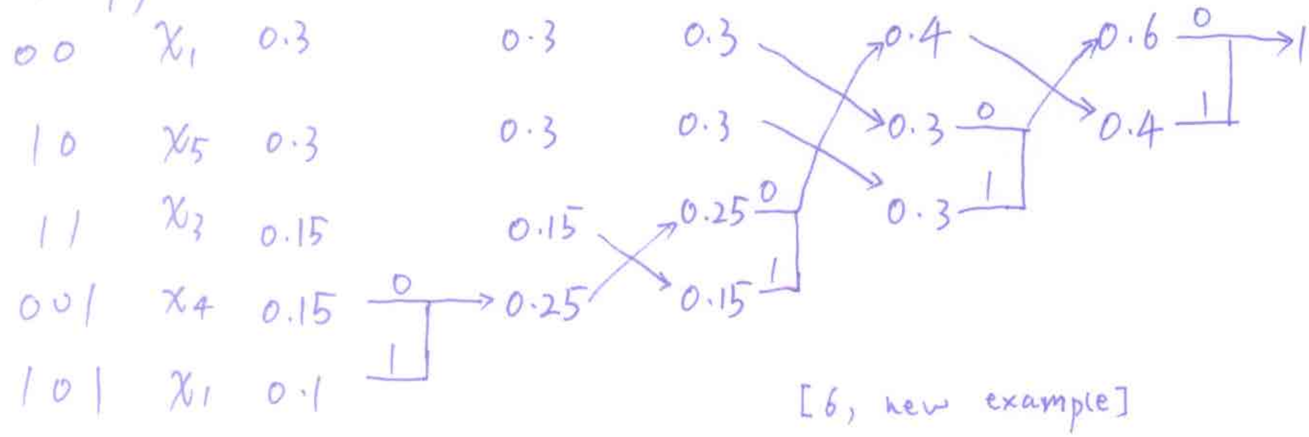
$$= -\left.\frac{e^{-t^2/2}}{\sqrt{2\pi}\,t}\right|_x^\infty - \int_x^\infty \frac{e^{-t^2/2}}{\sqrt{2\pi}\,t^2}\, dt$$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}\,x} - \int_x^\infty \frac{e^{-t^2/2}}{\sqrt{2\pi}\,t^2}\, dt$$

$$\le \frac{e^{-x^2/2}}{\sqrt{2\pi}\,x}$$ Since the second term is non negative.

[6, new theory]

4\. a) i)



[6, new example]

ii) Average code word length

$$\bar{L} = 2\times 0.3 + 2\times 0.3 + 2\times 0.15 + 3\times 0.15 + 3\times 0.1$$

$$= 2.25$$

Entropy

$$H(S) = -\sum p_i \log_2(p_i)$$

$$= 2\times(-0.3\times(\log_2(0.3)) + 2\times(-0.15\times \log_2(0.15))$$

$$+ (-0.1\times \log_2(0.1))$$

$$= 2.19$$

$$\eta = \frac{2.19}{2.25} = 97.3\%$$ [6, new application]

b) i)

$$G = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\text{row 4 + row 3}} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[\text{row}3]{\text{row}2+} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

[6, new application]

$$\xrightarrow[\text{row}4]{\text{row}1+\ \text{row}2+} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Systematic form

ii) $g(z)\,h(z) = z^7 + 1$

$h(z) = z^4 + z^3 + z^2 + 1$

[4, new example]

$$\begin{array}{r} z^4 + z^3 + z^2 + 1 \\ z^3+z^2+1 \,\big)\, \overline{z^7 + 1} \\ \underline{z^7 + z^6 + z^4} \\ z^6 + z^4 + 1 \\ \underline{z^6 + z^5 + z^3} \\ z^5 + z^4 + z^3 + 1 \\ \underline{z^5 + z^4 + z^2} \\ z^3 + z^2 + 1 \end{array}$$

iii) $$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$d_{min} = 3$ because the smallest number of dependent rows is 3.

[4, bookwork]

error detection $t = d_{min} - 1 = 2$

error correction $t = \frac{d_{min} - 1}{2} = 1$

iv) Hamming Bound $r = n - k \geq \log_2 \zeta(n,t)$

$$\zeta(n,t) = \sum_{i=0}^{t} \binom{n}{i} \underset{n=7}{\overset{t=1}{=\!=}} \binom{7}{0} + \binom{7}{1} = 8$$

$\log_2 \zeta(n,t) = 3$

$r = n - k = 3$

$\therefore$ Yes, it's a 'perfect' code in this sense.

[4, new application]